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# Exact Solutions of Some Nonlinear Equations

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We obtain exact solutions of three nonlinear diffusive equations and of the KdV–Burger equation by making an ansatz for the solution in each case.

## 1. INTRODUCTION

Methods of solving nonlinear partial differential equations are limited in number. Moreover, each of the methods, e.g., the inverse scattering method (Gardner *et al.*, 1967), Hirota's method (Hirota, 1971), the trace method (Wadati and Sawada, 1980), and the direct algebraic method (Hereman *et al.*, 1986), has some limitations. In this paper, we give exact solutions of four nonlinear equations in a rather simple way, using the methodology of Lan and Kelin (1990). These exact solutions are completely different in form from those of Ablowitz and Zeppetalle (1979), McKean (1970), and Reitz (1981), which are all perturbative solutions.

## 2. METHOD OF SOLUTION

### 2.1. Nagumo's Equation

For a large class of general one-dimensional single-component diffusive equations, a representative model is usually given by

$$\frac{\partial \phi(x, t)}{\partial t} = d \frac{\partial^2 \phi}{\partial x^2} + F(\phi) \tag{2.1}$$

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where $F(\phi)$ is some nonlinear function of $\phi$ and $d$ is the diffusion coefficient; of particular importance in equation (2.1) is the polynomial case

$$F(\phi) = \alpha\phi^m + \beta\phi^n$$

where $m$ and $n$ are integers. For example, $m=1$ and $n=2$ ($\alpha=-\beta=1$) gives the Fisher equation (Lakshmanan and Kaliappan, 1979); when $m=1$ and $n=3$, we have Nagumo's equation (McKean, 1970); and when $m=l+1$, $n=2l+1$ ($\alpha=\beta=K$) we have the Splading equation (Reitz, 1981). So Nagumo's equation takes the form

$$\phi_t - d\phi_{xx} = \phi - \phi^3 \tag{2.2}$$

We now look for traveling wave solutions of (2.2); that is, we assume

$$\phi(x,t) = \phi(x-\lambda t) = \phi(\xi) \tag{2.3}$$

where $\lambda$ is the velocity to be determined. Inserting (2.3) in (2.2), we get

$$-\lambda\phi_\xi - d\phi_{\xi\xi} = \phi - \phi^3 \tag{2.4}$$

With regard to equation (2.4), following the method of Lan and Kelin (1990), we make the following ansatz:

$$\phi = \sum_{i=0}^{m'} a_i(\tanh\mu\xi)^i \tag{2.5}$$

where the integers $m'$, $a_i$ ($i=1,\ldots,m'$), and $\mu$ are parameters to be determined. The requirement that that highest power of the function $\tanh\mu\xi$ for the nonlinear term $\phi^3$ and that for the derivative term $\phi_{\xi\xi}$ must be equal gives the following relation:

$$3m' = m' + 2$$

Thus we get $m'=1$ and equation (2.5) can be written as

$$\phi = a_0 + a_1\tanh\mu\xi \tag{2.6}$$

Inserting equation (2.6) into (2.4), we get the following parametric equations upon equating the same powers of $\tanh\mu\xi$:

$$-\lambda a_1\mu = a_0 - a_0^3 \tag{2.7a}$$

$$-2a_1 d\mu^2 = a_1 - 3a_0^2 a_1 \tag{2.7b}$$

$$\lambda a_1\mu = -3a_0 a_1^2 \tag{2.7c}$$

$$2a_1 d\mu^2 = -a_1^3 \tag{2.7d}$$

From (2.7), we get

$$a_0 = \pm(2/5)^{1/2}$$

$$a_1 = \pm(1/5)^{1/2}$$

$$\mu = \pm(1/10d)^{1/2}$$

$$\lambda = \pm(36d/5)^{1/2}$$

Thus we obtain exact solutions of equation (2.2) representing waves in both directions.

## 2.2. Fisher's Equation

Fisher's equation reads

$$\phi_t - d\phi_{xx} = \phi - \phi^2 \tag{2.8}$$

For the traveling wave solution, we assume

$$\phi(x, t) = \phi(x - \lambda t) = \phi(\xi) \tag{2.9}$$

where $\lambda$ is the velocity to be determined.

Inserting (2.9) into (2.8), we get

$$-\lambda\phi_\xi - d\phi_{\xi\xi} = \phi - \phi^2 \tag{2.10}$$

With regard to equation (2.10), we again make the ansatz (2.5) and here the requirement that the highest power of the function ($\tanh \mu\xi$) for the nonlinear term $\phi^2$ and that for the derivative term $\phi_{\xi\xi}$ must be equal gives the following relation:

$$2m' = m' + 2$$

Thus, here we get $m' = 2$ and equation (2.5) now takes the form

$$\phi = a_0 + a_1 \tanh \mu\xi + a_2 \tanh^2 \mu\xi \tag{2.11}$$

Inserting equation (2.11) into (2.10), we get the following parametric equations upon equating the same powers of ($\tanh \mu\xi$):

$$-\lambda a_1\mu - 2a_2 d\mu^2 = a_0 - a_0^2 \tag{2.12a}$$

$$-2a_2\mu\lambda + 2a_1 d\mu^2 = a_1 - 2a_0a_1 \tag{2.12b}$$

$$\lambda a_1\mu + 8a_2 d\mu^2 = a_2 - a_1^2 - 2a_0a_2 \tag{2.12c}$$

$$2\mu a_2\lambda - 2a_1 d\mu^2 = -2a_1a_2 \tag{2.12d}$$

$$-6a_2 d\mu^2 = -a_2^2 \tag{2.12e}$$

From equations (2.12), we get

$$a_0 = 1/4$$

$$a_1 = \pm 1/2$$

$$a_2 = 1/4$$

$$\mu = \pm(1/24d)^{1/2}$$

$$\lambda = \pm 5(d/6)^{1/2}$$

Thus we obtain exact solutions of equation (2.8) representing waves capable of moving in both directions.

### 2.3. Splading Equation

The Splading equation has the form

$$\phi_t - d\phi_{xx} = K[\phi^m + \phi^n] \tag{2.13}$$

where $m = l+1$ and $n = 2l+1$. In this case, the requirement that the highest power of ($\tanh \mu\xi$) for the nonlinear term $\phi^{2l+1}$ equal that for the derivative term $\phi_{\xi\xi}$ gives

$$m' + 2 = (2l+1)m'$$

Thus we get $m' = 1/l$. Now, since $m'$, $m$, and $n$ are integers, the Splading equation can be solved by the technique used here for $l = 1$ only. We give the solution, obtained as before:

$$\phi = a_0 + a_1 \tanh \mu\xi \tag{2.14}$$

where

$$a_0 = \left[-1 \pm \frac{\lambda}{(-2Kd)^{1/2}}\right] \Big/ 3 \tag{2.15a}$$

$$a_1 = \pm\left(\frac{-2d\mu^2}{K}\right)^{1/2} \tag{2.15b}$$

$$= \left(\frac{-K}{6d}\right)^{1/2}\left[1 \mp \frac{\lambda}{(-2Kd)^{1/2}}\right] \tag{2.15c}$$

together with one constraint relating $\lambda$, $\mu$, $K$, $d$, and $a_0$:

$$\mp\lambda\left(\frac{-2d}{K}\right)^{1/2}\mu^2 = Ka_0^2(1 + a_0)$$

### 2.4. KdV–Burger Equation

The KdV equation perturbed by a Burger-like dissipative term reads

$$u_t + \mu u u_x + \rho u_{xxx} - \nu u_{xx} = 0 \tag{2.16}$$

When the dissipation is dominant, equation (2.16) is known to possess shocklike solitary waves, while in the pure dispersive limit ($\nu=0$), equation (2.16) admits KdV solutions (Reinisch *et al.*, 1978). Using Lie's method of continuous transformation groups, Lakshmanan and Kaliappan (1979) deduced a class of invariant solutions which are in general not of Painleve type. Equations (11) and (12) of Lakshmanan and Kaliappan (1979) take the following forms when $\beta=0$:

$$u=f(\zeta), \qquad \zeta=\left[x-\left(\frac{\delta}{\alpha}\right)t\right] \tag{2.17}$$

Substituting (2.17) into (2.16) and integrating once, we get

$$f_{\zeta\zeta}-\left(\frac{\nu}{\rho}\right)f_\zeta-\left(\frac{\mu}{2\rho}\right)f^2-\left(\frac{\delta}{\alpha\rho}\right)f+C_0=0 \tag{2.18}$$

where $C_0$ is the constant of integration.

We now make the following scale transformation:

$$\phi=K_1 f+K_2 \tag{2.19a}$$

$$\xi=K_3+K_4 \tag{2.19b}$$

Equation (2.18) is then reduced to the form

$$\phi_{\xi\xi}-\frac{\nu}{\rho}\left(\frac{\alpha^2\rho^2}{\delta^2+2\alpha^2\mu\rho C_0}\right)^{-1/4}\phi_\xi+\phi-\phi^2=0$$

This is of the form of equation (2.10) except

$$\lambda=-\frac{\nu}{\rho}\left(\frac{\alpha^2\rho^2}{\delta^2+2\alpha^2\mu\rho C_0}\right)^{-1/4}, \qquad d=1$$

From this we infer that the KdV–Burger equation has the same traveling wave solutions as in the case of Fisher's equation with arbitrary wave speed

$$\frac{\delta}{\alpha}=\pm\left(\frac{36\nu^2}{625}-2\mu\rho C_0\right)^{1/2}$$

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